WILEY

RESEARCH ARTICLE

Generating OWA weights using truncated distributions

Maxime Lenormand

Irstea, UMR TETIS, Montpellier, France

Correspondence
Irstea, UMR TETIS, 500 rue JF Breton, FR-34093 Montpellier, France.
Email: maxime.lenormand@irstea.fr

ABSTRACT

Ordered weighted averaging (OWA) operators have been widely used in decision making these past few years. An important issue facing the OWA operators' users is the determination of the OWA weights. This paper introduces an OWA determination method based on truncated distributions that enable intuitive generation of OWA weights according to a certain level of risk and trade-off. These two dimensions are represented by the two first moments of the truncated distribution. We illustrate our approach with the well-known normal distribution and the definition of a continuous parabolic decision-strategy space. We finally study the impact of the number of criteria on the results.

KEYWORDS

OWA operators, probability density function, Risk and Trade-off, truncated distribution

1 | INTRODUCTION

The ordered weighted averaging (OWA) operator was first introduced by Yager[1] in 1998 to address the problem of aggregating a set of criteria functions to form an overall decision function. This seminal paper has shed some light on the recurrent problem of aggregation shared by many disciplines and used in a wide range of applications.[2] In particular, OWA operators have been extensively used in multiple criteria decision analysis to provide support in complex decision-making situations. OWA operators have notably been applied in GIS environments for the production of land-use suitability maps[3] with applications in land-use planning and management such as landslide susceptibility mapping,[4] mapping of wilderness,[5] ecological capability assessment,[6] disease susceptibility mapping,[7] or soil fertility evaluation.[8] OWA operators are indeed particularly interesting in decision analysis, because they

offer the possibility to incorporate a certain level of risk and trade-off in decision-making processes.[9,10] These two concepts are inextricably connected to the distribution of OWA weights, the form of which implies the satisfaction of one or several criteria, reflecting a certain risk inclination/aversion and willingness to compromise.

An important issue regarding the manipulation of OWA operators is to determine the weights. The result of aggregation is indeed highly dependent of the vector of OWA weights used to aggregate the criteria. OWA weights generation is an active research topic, and numerous approaches have been proposed during the past few years.[11] First, we need to make a distinction between OWA weights determination methods based on external information and methods focusing on the property of the weights distribution itself. With the first type of methods, the OWA weights can be fitted to empirical data,[12,13] function of the criteria values,[14,15] or based on preference relations.[16] Although these methods are very interesting and closely related to real application problems, they are usually case dependent and rely heavily on the quality of data sources. Therefore, in this paper, we will focus on the latter type of methods, the results of which essentially depend on the structure and properties of the OWA weight vector. These methods can be divided into two main categories: the optimization-based methods and the function-based methods. Optimization-based methods usually rely on the concept of *orness*,[1] which measure the "orlike" level of OWA operators. It can be seen as a measure of risk in decision analysis.[9,10] The objective of the optimization-based methods is basically to maximize the dispersion of the weights (i.e., trade-off between criteria) for a given level of *orness* (i.e. risk).[17–22] The second class of methods proposes to generate OWA weights using continuous mathematical functions whose shape will determine the OWA weights distribution after a discretization step. The first functions that were proposed relied on linguistic quantifiers to guide the aggregation process.[12,14] Different forms of quantifiers and their relationships with *orness* and maximum dispersion has been studied in the past few years.[23–26] Yager also proposed a particular class of function called "stress function" that allows greater emphasis on particular criteria.[27] Another interesting class of functions rely on probability theory. Xu proposed in 2005 the use of normal distributions to generate OWA weights.[28] This idea was taken up by Yager in a paper dealing with centered OWA operator[29] and generalized by Sadiq and Tesfamarian to other families of probability density functions (PDF).[30]

It is interesting to note that none of these methods allow for a full exploration of the decision-strategy space defined by a certain level of risk ("orlike" level) but also a certain level of trade-off (dispersion).[9] Measuring the impact of the level of risk and trade-off on a decision represents, however, a crucial step in a decision-making process. It is therefore important to propose methods that automatically generates OWA weights according to a certain level of risk and trade-off allowing for a systematic and rigorous sensitivity analysis assessing the impact of these values on the final output. Therefore, we fundamentally believe that the generation of OWA weights requires a clear understanding of existing relationships between weight distribution, risk, and trade-off. To the best of our knowledge, there is no method that allows for an automatic generation of order weights according to a certain level of risk and trade-off.

In this article, we tackle this issue by disentangling the relationships between OWA weights, risk, and trade-off using truncated distributions. We will first describe the basic concept of OWA operators and formally introduce the concepts of risk and trade-off. We will then explain in detail the method that we developed to generate OWA weights according to a certain level of risk and trade-off. In particular, we will focus on the truncated normal distribution and the definition of a continuous parabolic decision-strategy space. Finally, we will explore the influence of the number of criteria on the results.

2 | OWA OPERATOR, RISK, AND TRADE-OFF

Let us consider a collection of n objects $(x_1, ..., x_n)$ representing preference values associated with a decision to take on a particular issue. The problem is to aggregate the n objects together to obtain an unique object to enable a production decision. The considered objects may have several dimensions or even be weighted, but the decision will be always made on an element-by-element basis; so without loss of generality, let us consider that the x_i are real numbers, hereafter, called criteria.

To solve this problem, Yager in 1988 introduced a new family of aggregation techniques called OWA operators.[1] Formally, the OWA operator $O_w : \mathbb{R}^n \rightarrow \mathbb{R}$ associates a set of order weights $w = (w_1, ..., w_n)$ such that $\sum_{i=1}^{n} w_i = 1$ to our collection of n criteria as follows:

$$O_w(x_1, ..., x_n) = \sum_{i=1}^{n} w_i x_{(i)} \tag{1}$$

where $x_{(i)}$ is the ith lowest value in $(x_1, ..., x_n)$. The first-order weight w_1 is therefore assigned to the criterion with the lowest value, the second-order weight w_2 to the second lowest criterion, etc. Two dimensions are usually associated with w: risk and trade-off. The first dimension characterizes the level of risk in the aggregation process, which positions the values obtained with the aggregation operator O_w on a continuum between the *minimum* and the *maximum* x_i values. The level of risk can be measured from w using the concept of *orness* or *andness* as follows:

$$\begin{cases} andness(w) = \frac{1}{n-1} \sum_{i=1}^{n} w_i(n-i) \\ orness(w) = 1 - andness(w) \end{cases} \tag{2}$$

Orness and *andness* are two complementary values that reflect an attitude toward risk in decision making. An *andness* value equal to 1 (or *orness* equal to 0) gives full weight to the *minimum* x_i value (intersection, AND operator). It represents a risk-aversion position where the vector of OWA order weights w becomes $(1, 0, ..., 0)$. On the other extreme, an *orness* equal to 1 (or *andness* equal to 0) gives full weight to the *maximum* x_i value (union, OR operator). It represents the most risk-taking attitude where the vector of OWA order weights w is equal to $(0, 0, ..., 1)$.

The second dimension represents the level of trade-off between criteria. It can be seen as a measure of dispersion over the OWA order weights. The concept of dispersion in OWA was first introduced by Yager in 1988. Similar to the entropy of Shannon, the measure of dispersion can be computed as follows:
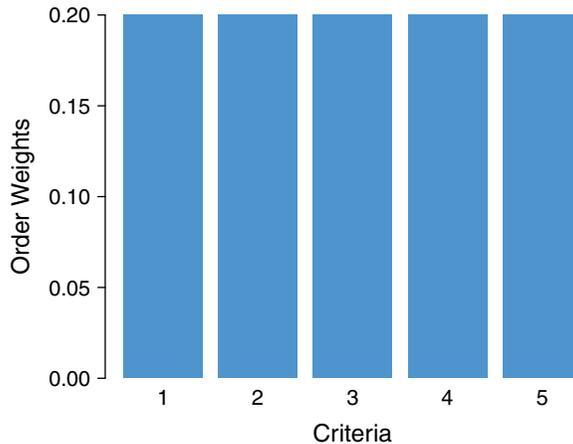
$$Disp(w) = \frac{- \sum_{i=1}^{n} w_i \cdot \log(w_i)}{\log(n)} \tag{3}$$

This measure is comprised between 0 and 1, reaching its maximum when $w_i = 1/n$. Another measure of trade-off was, proposed by Jiang and Eastman,[9] computed as a distance to the uniform distribution:

$$trade-off(w) = 1 - \sqrt{\frac{n \sum_{i=1}^{n} (w_i - 1/n)^2}{n-1}} \tag{4}$$
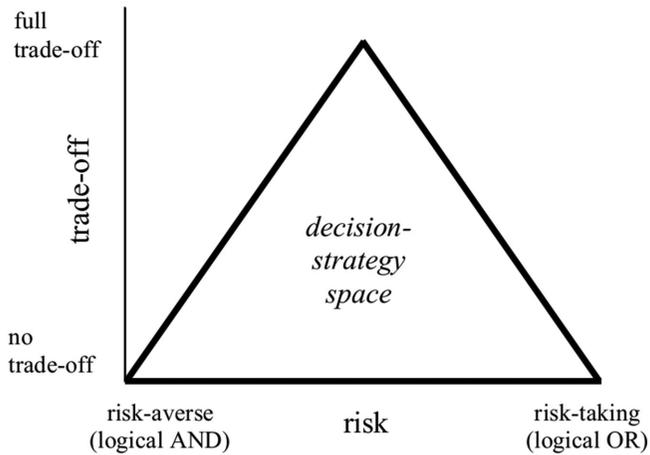
FIGURE 1 Triangular decision-strategy space defined by the dimension of risk and trade-off. The figure is taken from Ref. 10

FIGURE 2 Distribution of order weights for five criteria with average level of risk ($\alpha_w = 0.5$) and full trade-off ($\delta_w = 1$)

Various measures could be considered to assess the level of risk and trade-off associated with a vector of OWA order weights w. Without loss of generality, they can be expressed as a couple of values $(\alpha_w, \delta_w) \in [0, 1]^2$. It is, however, important to note that these two dimensions are not independent; certain couple of values are inconsistent. It is indeed not possible to obtain a vector w, which exhibits at the same time a low or high level of risk and a high level of trade-off. As shown in Figure 1, this two-dimensional space can be approximated by a triangle.

The three vertices of the triangle represents the three main OWA operators: minimum ($\alpha_w = 0$, $\delta_w = 0$), maximum ($\alpha_w = 1$, $\delta_w = 0$), and average ($\alpha_w = 0.5$, $\delta_w = 1$). The average operator corresponds to a weighted linear combination of criteria with a middle level of risk ($\alpha_w = 0.5$) and a full trade-off ($\delta_w = 1$). An example of such distribution of order weights for five criteria is presented in Figure 2. Except for some trivial cases, like the ones described above, it is not formally established, in practice, how to generate order weights according to a certain level of risk and trade-off.
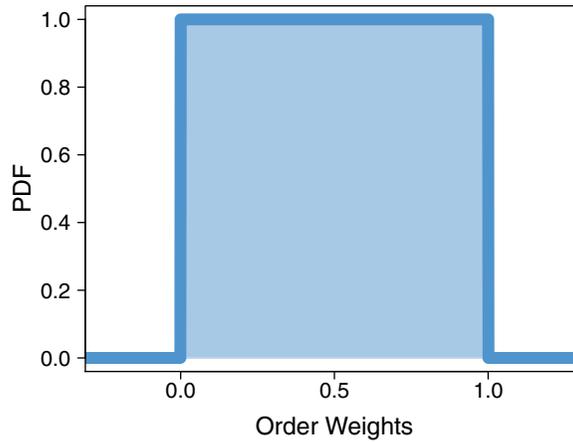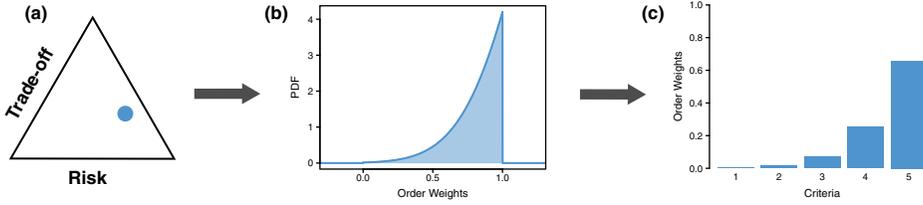
FIGURE 3 Continuous distribution of order weights with average level of risk ($\alpha_w = 0.5$) and full trade-off ($\delta_w = 1$)

3 | GENERATION OF OWA WEIGHTS USING TRUNCATED DISTRIBUTIONS

The generation of OWA weights generation can be investigated through the existing relationships between order weights, risk, and trade-off. Indeed, a formal description of the relationships between these concepts will permit us to not only better understand the link between risk and trade-off (i.e., decision-strategy space), but to propose a way to generate automatically OWA order weights from any risk and trade-off values in this triangular decision-strategy space. Here, we tackle this problem by proposing an approach relying on PDF to generate a vector of order weights w according to a couple of risk and trade-off values (α_w, δ_w) comprised between 0 and 1. Indeed, as already pointed out in several studies, OWA order weights can be viewed as a discrete PDF represented by a vector of real numbers comprised between 0 and 1 that sum to 1.[28,30] Therefore, an OWA weights distribution can be modeled asymptotically (when $n \to \infty$) as a continuous probability density distribution f_w. An example is given in Figure 3, in which the continuous version of the distribution of order weights presented in Figure 2 is displayed.

It is important to note that, unlike other approaches, the domain of f_w is bounded and, for simplicity purposes, is set to [0, 1]. This choice of a bounded domain for f_w offers several advantages. The main one is to consider truncated distributions characterized by bounded averages and standard deviations. The average μ_w is comprised between 0 and 1, and the standard deviation σ_w is upper-bounded by $\frac{1}{2\sqrt{3}}$, standard deviation of a uniform distribution $\mathcal{U}_{[0,1]}$. Average and standard deviation represent the two first moments of a PDF, and these two statistics have several properties in common with the risk and trade-off measures associated with OWA order weights. As described in Equation 2, *andness* and *orness* are governed by the skewness of w.[9] In our case, the level of asymmetry of f_w depends on μ_w, positive skew when $\mu_w < 0.5$ and negative skew when $\mu_w > 0.5$. In the same way, Equation 3 or 4 shows that the level of trade-off is controlled by the degree of dispersion in the order weights.[9] In a probability distribution, the degree of dispersion is represented by the standard deviation σ_w. We also observe the same relationship between μ_w and σ_w and α_w and δ_w. In both cases, high levels of asymmetry are incompatible with a high level of dispersion and the maximum dispersion can be reached when the average equal the median. The main difference between the two couples of metrics is that the upper

FIGURE 4 Automatic generation of OWA weights according to a certain level of risk and trade-off based on truncated probability distribution. (a) Choose a certain level of risk and trade-off. (b) Generation of a continuous OWA order weights distribution according to a certain level of risk and trade-off. The continuous distribution take the form of a PDF of a truncated probability distribution. (c) Discretization of the PDF to obtain n order weights

bound for dispersion is different. Thus, we define a relationship between risk and trade-off and average and standard deviation as follows:

$$\begin{cases} \alpha_w = \mu_w \\ \delta_w = 2\sqrt{3}\,\sigma_w \end{cases} \tag{5}$$

Hence, any truncated distribution restricted on the domain $[0, 1]$ satisfying Equation 5 can be seen as a continuous distribution of OWA order weights with level of risk and trade-off equal to α_w and δ_w, respectively. Using truncated distribution also offers a simple way to describe how the OWA order weights are generated from f_w. Equation 6 is indeed required to discretize f_w and obtain a vector of n OWA order weights.

$$w_i = \frac{f_w\left(\frac{i-1}{n-1}\right)}{\sum\limits_{k=1}^{n} f_w\left(\frac{k-1}{n-1}\right)}\ , \ \forall i \in \|[1, n]\| \tag{6}$$

Nevertheless, it is important to ensure that the denominator in Equation 6 is not equal to 0. In particular, when $\delta_w = \sigma_w = 0$, f_w is represented by a generalized PDF $f_w(x) = \delta(x - \alpha_w)$ using the Dirac delta function. In this particular case, the weight with normalized index $\frac{i-1}{n-1}$ nearest to α_w is equal to 1 and all the others are set to 0.

Hence, we showed that truncated distributions can be used to generate automatically OWA weights from any risk and trade-off values in this triangular decision-strategy space (Figure 4).

4 | CONTINUOUS PARABOLIC DECISION-STRATEGY SPACE

In practice, not all families of probability distribution are suitable to generate meaningful OWA weights. In this study, we focused on the truncated normal distribution $\mathcal{N}_{[0,1]}(\mu_w, \sigma_w)$ restricted on the domain $[0, 1]$. μ_w and σ_w represent the average and standard deviation computed after truncation, respectively. As explained in the preceding section, for a certain level of risk $\alpha_w \in [0, 1]$, and trade-off $\delta_w \in [0, 1]$, one can model a continuous OWA order weight distribution f_w using a truncated normal distribution $\mathcal{N}_{[0,1]}(\mu_w, \sigma_w)$ satisfying Equation 5.

However, an important issue that we need to address concerns the generation of the truncated normal distribution itself. In fact, any normal distribution with average μ and standard deviation σ lying within

the interval [0, 1] is a truncated normal distributions of average μ_w and standard deviation σ_w. Its PDF is given by

$$f(x) = \frac{\phi\left(\frac{x-\mu}{\sigma}\right)}{\sigma\left(\Phi\left(\frac{1-\mu}{\sigma}\right) - \Phi\left(-\frac{\mu}{\sigma}\right)\right)} \tag{7}$$

when $x \in [0, 1]$ and 0 otherwise. ϕ is the PDF of the standard normal distribution, and Φ is its cumulative distribution function. The mean and standard deviation of the truncated distribution on the domain [0, 1] can be computed with the formula displayed in Equations 8 and 9, respectively:

$$\mu_w = \mu + \sigma\frac{\phi\left(-\frac{\mu}{\sigma}\right) - \phi\left(\frac{1-\mu}{\sigma}\right)}{\Phi\left(\frac{1-\mu}{\sigma}\right) - \Phi\left(-\frac{\mu}{\sigma}\right)} \tag{8}$$

$$\sigma_w = \sigma\sqrt{1 + \frac{\frac{1-\mu}{\sigma}\phi\left(\frac{1-\mu}{\sigma}\right) + \frac{\mu}{\sigma}\phi\left(-\frac{\mu}{\sigma}\right)}{\Phi\left(\frac{1-\mu}{\sigma}\right) - \Phi\left(-\frac{\mu}{\sigma}\right)} - \left(\frac{\phi\left(-\frac{\mu}{\sigma}\right) - \phi\left(\frac{1-\mu}{\sigma}\right)}{\Phi\left(-\frac{\mu}{\sigma}\right) - \Phi\left(\frac{1-\mu}{\sigma}\right)}\right)^2} \tag{9}$$

Consequently, to generate a truncated normal distribution $\mathcal{N}_{[0,1]}(\mu_w, \sigma_w)$, we first need to identify the normal distribution $\mathcal{N}(\mu, \sigma)$ whose restriction on the domain [0, 1] will give us an average of μ_w and a standard deviation equal to σ_w. Although it exists a clear relationship $(\mu, \sigma) \rightarrow (\mu_w, \sigma_w)$ allowing for the generation of a truncated normal distribution f_w from any normal distribution of parameter μ and σ, the opposite is not true. Indeed, as previously explained, the more μ_w deviates from 0.5, the smaller the maximal value that σ_w can take, thus excluding a wide range of combinations.

To overcome this issue, we have decided here to favor a numerical approach. Let us consider a couple of risk and trade-off values (α_w, δ_w) comprised between 0 and 1. The associated average and standard deviation (μ_w, σ_w) can be easily derived from Equation 5. We then need to identify the couple of values (μ, σ) whose "truncated" average and standard deviation $(\hat{\mu}_w, \hat{\sigma}_w)$ on the domain [0, 1] are the closest from (μ_w, σ_w). To do so, we used an optimization algorithm,[31] which seeks to find the optimal combination of μ and σ values minimizing the quadratic distance d between the expected (μ_w, σ_w) values and the simulated ones (Equation 10).

$$d = \sqrt{(\mu_w - \hat{\mu}_w)^2 + (\sigma_w - \hat{\sigma}_w)^2} \tag{10}$$

At the end of the process, we obtain a candidate PDF f_w, the validity of which is evaluated with d.

We assume that if d is higher than a predetermined threshold ϵ, then no suitable PDF can be found with these risk and trade-off values. A simple way to calibrate the threshold value ϵ is to investigate its impact on the fraction of rejected estimations such as $d \geq \epsilon$. For that purpose, we drawn at random 10,000 couple of risk and trade-off values using a Latin hypercube sampling. Then, we applied the process described above to each element of our sample to quantify the distance d associated with each of them. As it can be observed in Figure 5, the fraction of rejected estimations is high for very small values of ϵ. Then, it starts to decrease when ϵ increases, until it reaches a plateau between 10^{-8} and 10^{-3}. This plateau represents a gap between rejected and accepted estimations. Figure 6 shows the two-dimensional representation of the rejected and accepted estimations with $\epsilon = 10^{-8}$ (beginning of the plateau) as a function of the level of risk and trade-off. It is interesting to observe that the frontier
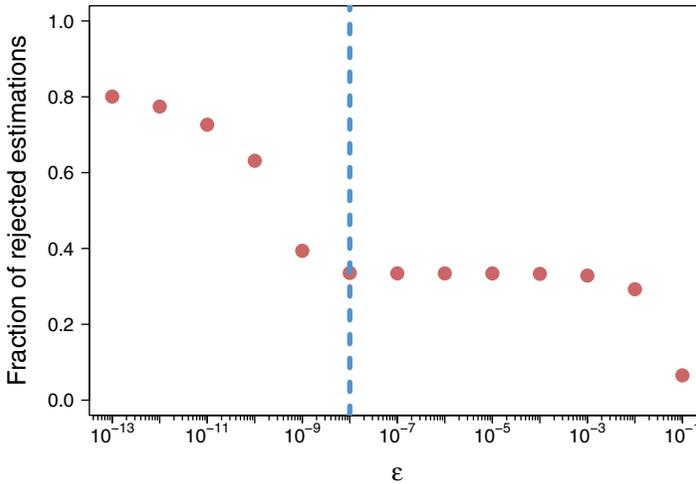
FIGURE 5 Evolution of the fraction of rejected estimations as a function of ϵ. The blue dashed line represent the optimal ϵ value
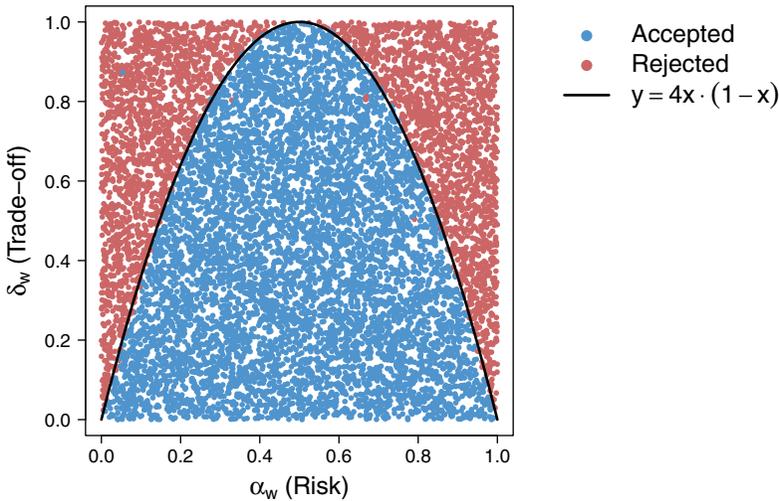

FIGURE 6 Representation of the continuous parabolic decision-strategy space defined by the dimension of risk and trade-off based on truncated normal distribution. The blue dots represent the couple of risk and trade-off value for which $d < \epsilon$. The red dots represent the couple of risk and trade-off value for which $d \geq \epsilon$. The black line represents the frontier between the two regions approximated with a parabola

between rejected and accepted estimations coincides with a parabola whose equation is $y = 4x(1 - x)$. It is therefore possible to formally describe the relationship between risk and trade-off and define a continuous parabolic decision-strategy space to automatically generates OWA weights.

5 | INFLUENCE OF THE NUMBER OF CRITERIA

In this section, we investigate the influence of the number of criteria, n, on the properties of the weight distribution generated with the truncated normal distribution for a given level of risk and trade-off. It
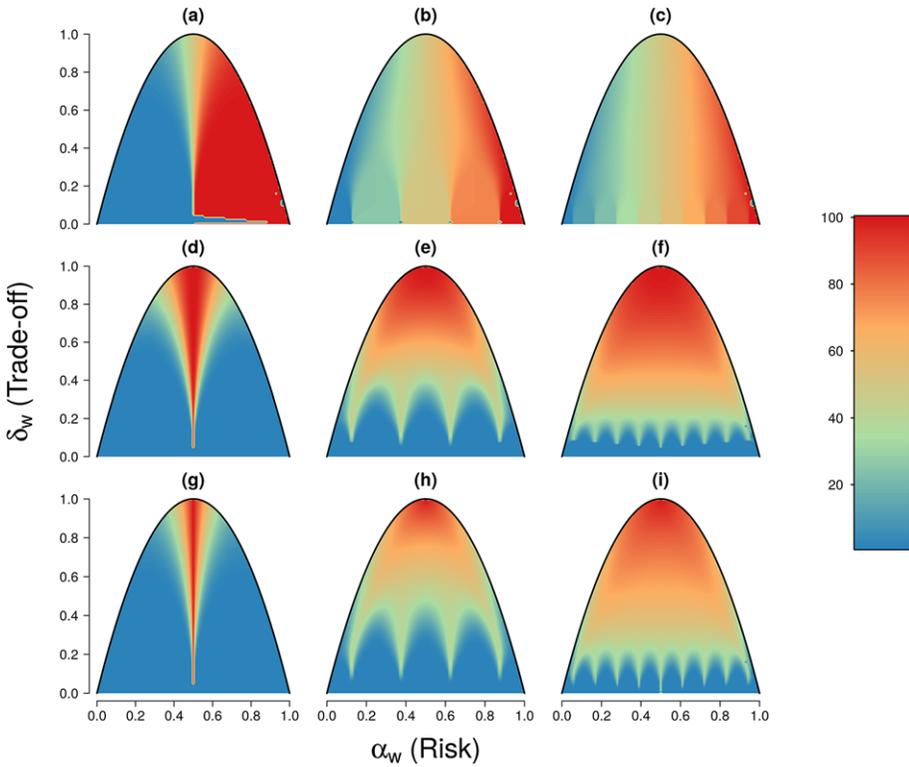
FIGURE 7 Influence of the number of criteria on the OWA weights' properties generated with the truncated normal distribution for a given level of risk and trade-off. From the left to the right, the results obtained with two criteria ((a), (d), and (g)), five criteria ((b), (e), and (h)), and ten criteria ((c), (f), and (i)) are displayed. From the top to the bottom, the properties of the OWA weight distribution are measured with the *orness* as defined in Equation 2 ((a)–(c)), the dispersion as defined in Equation 3 ((d)–(f)) and trade-off as defined in Equation 4 ((g)–(i))

is indeed important to keep in mind that the discretization step (Equation 6) has a non-negligible effect on the shape of the weight distribution, particularly when the number of criteria is small. To do so, we relied on three well-known metrics described in the first part of the paper to evaluate the properties of the OWA weight distribution: the *orness*, the *Disp* and the *trade − off* as defined in Equations 2, 3, and 4, respectively. The evolution of these metrics as a function of α_w and δ_w for different values of n (2, 5, and 10) are displayed in Figure 7. It can be observed that the weight distributions are less sensitive to variations in α_w and δ_w values when the number of criteria decreases. This is particularly true for small δ_w values. Indeed, when n decreases, the level of trade-off required to find a compromise between at least two criteria ($w_i < 1, \forall i$) becomes higher. This phenomenon is amplified by the level of risk, α_n, the value of which may reflect a certain willingness to compromise between two consecutive criteria i and $i + 1$ according to its position in the range bin $[\frac{i-1}{n-1}, \frac{i-1}{n-1}]$. Although the discretization step has very strong effect on the weight generation for very small numbers of criteria, $n = 2$ or 3, it becomes less and less significant as n increases. Besides, it is also worth noting that the dispersion associated with w obtained for a given value of α_w and δ_w tend to increase when n increases. This is a property of the dispersion as defined in Equations 3 and 4, which values tend to be closer to 1 as $n \rightarrow \infty.$[3]

6 | DISCUSSION

In summary, we introduced a new method to determine OWA weights. The method is based on PDF of truncated distributions. It allows for an automatic generation of OWA weights according to a certain level of risk and trade-off based on the two first moments of the probability distribution. The main advantage of our method is to provide a rigorous framework for a full exploration of the decision-strategy space (Figure 4). Moreover, the proposed method enables to conduct systematic sensitivity and uncertainty analysis on multicriteria decision analysis and makes it possible to better assess the impact of the level of risk and trade-off on the final decision.

In this work, we focused on the truncated normal distribution and unveiled the relationship between risk and trade-off by identifying a well-defined continuous parabolic decision-strategy space. We have then examined the effect of the discretization step on the weight distribution. Unsurprisingly, we showed that the sensitivity to variations in risk and trade-off values with the number of criteria. An inevitable direction for further studies will be to extent this analysis to other families of PDFs.

We are especially interested in proposing a method accessible to non-specialist policymakers. This is why we decided to focus on OWA weights generation through PDF. They have indeed the clear advantage of simplicity while relying on well-established statistical properties. Finally, the software package to generate OWA using the approach described in the paper along with an interactive web application for visualizing the results can be downloaded from https://github.com/maximelenormand/OWA-weights-generator.

REFERENCES

1. Yager RR. On ordered weighted averaging aggregation operators in multicriteria decisionmaking. *IEEE Ttans Syst Man Cyb*. 1988;18(1):183–190.

2. Yager RR, Kacprzyk J, Beliakov G. Recent Developments in the Ordered Weighted Averaging Operators: Theory and Practice. Berlin, Germany: Springer; 2011.

3. Malczewski J. Ordered weighted averaging with fuzzy quantifiers: GIS-based multicriteria evaluation for land-use suitability analysis. *Int J Appl Earth Observat Geoinform*. 2006;8(4):270–277.

4. Feizizadeh B, Jankowski P, Blaschke T. A GIS based spatially-explicit sensitivity and uncertainty analysis approach for multi-criteria decision analysis. *Comput Geosci*. 2014;64:81–95.

5. Comber A, Carver S, Fritz S, McMorran R, Washtell J, Fisher P. Different methods, different wilds: evaluating alternative mappings of wildness using fuzzy MCE and Dempster–Shafer MCE. *Comput Environ Urban Syst*. 2010;34(2):142–152.

6. Ferretti V, Pomarico S. Ecological land suitability analysis through spatial indicators: an application of the analytic network process technique and ordered weighted average approach. *Ecol Indic*. 2013;34:507–519.

7. Dong S. Comparisons between different multi-criteria decision analysis techniques for disease susceptibility mapping. Student thesis series INES. 2016.

8. Mokarram M, Hojati M. Using ordered weight averaging (OWA) aggregation for multi-criteria soil fertility evaluation by GIS (case study: Southeast Iran). *Comput Electron Agric*. 2017;132:1–13.

9. Jiang H, Eastman JR. Application of fuzzy measures in multi-criteria evaluation in GIS. *Int J Geogr Inform Sci*. 2000;14(2):173–184.

10. Drobne S, Lisec A. Multi-attribute decision analysis in GIS: weighted linear combination and ordered weighted averaging. *Informatica*. 2009;33(4):459–474.

11. Liu X. A review of the OWA determination methods: Classification and some extensions. In: Yager RR, Kacprzyk J, Beliakov G, eds. Recent Developments in the Ordered Weighted Averaging Operators: Theory and Practice, number 265 in Studies in Fuzziness and Soft Computing. Berlin, Germany: Springer; 2011:49–90.

12. Yager RR. Quantifier guided aggregation using OWA operators. *Int J Intell Syst*. 1996;11(1):49–73.

13. Beliakov G, Mesiar R, Valaskova L. Fitting generated aggregation operators to empirical data. *Int J Uncertain Fuzz*. 2004;12(02):219–236.

14. Yager RR. Families of OWA operators. *Fuzzy Sets Syst*. 1993;59(2):125–148.

15. Xu Z. Dependent OWA operators. In: Modeling Decisions for Artificial Intelligence. Berlin, Germany: Springer; 2006:172–178.

16. Ahn BS. Preference relation approach for obtaining OWA operators weights. *Int J Approx Reason*. 2008;47(2):166–178.

17. O'Hagan M. Aggregating template or rule antecedents in real-time expert systems with fuzzy set logic. In: *Twenty-Second Asilomar Conference on Signals, Systems and Computers*, volume 2. 1988:681–689.

18. Yager RR, Filev DP. Parameterized and-like and or-like owa operators. *Int J Gen Syst*. 1994;22(3):297–316.

19. Filev D, Yager RR. Analytic properties of maximum entropy OWA operators. *Inform Sci*. 1995;85(1):11–27.

20. Filev D, Yager RR. On the issue of obtaining OWA operator weights. *Fuzzy Sets Syst*. 1998;94(2):157–169.

21. Fullér R, Majlender P. An analytic approach for obtaining maximal entropy OWA operator weights. *Fuzzy Sets Syst*. 2001;124(1):53–57.

22. Liu X. Models to determine parameterized ordered weighted averaging operators using optimization criteria. *Inform Sci*. 2012;190:27–55.

23. Liu X. On the properties of equidifferent RIM quantifier with generating function. *Int J Gen Syst*. 2005;34(5):579–594.

24. Liu X. Some properties of the weighted OWA operator. *IEEE T Syst Man Cybernet B*. 2006;36(1):118–127.

25. Liu X, Da Q. On the properties of regular increasing monotone (RIM) quantifiers with maximum entropy†. *Int J Gen Syst*. 2008;37(2):167–179.

26. Liu X, Han S. Orness and parameterized RIM quantifier aggregation with OWA operators: a summary. *Int J Approx Reason*. 2008;48(1):77–97.

27. Yager RR. Using stress functions to obtain OWA operators. *IEEE Trans Fuzzy Syst*. 2007;15(6):1122–1129.

28. Xu Z. An overview of methods for determining OWA weights. *Int J Intell Syst*. 2005;20(8):843–865.

29. Yager RR. Centered OWA operators. *Soft Comput*. 2007;11(7):631–639.

30. Sadiq R, Tesfamariam S. Probability density functions based weights for ordered weighted averaging (OWA) operators: An example of water quality indices. *Eur J Oper Res*. 2007;182(3):1350–1368.

31. Nelder JA, Mead R. A simplex method for function minimization. *Comput J*. 1965;7(4):308–313.